

String solutions in $AdS_3 \times S^3 \times S^3 \times S^1$ with B -field

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Abstract

We consider strings living in $AdS_3 \times S^3 \times S^3 \times S^1$ with nonzero B -field. By using specific ansatz for the string embedding, we obtain a class of solutions corresponding to strings moving in the whole ten dimensional space-time. For the AdS_3 subspace, these solutions are given in terms of incomplete elliptic integrals. For the two three-spheres, they are expressed in terms of Lauricella hypergeometric functions of many variables. The conserved charges, i.e. the string energy, spin and angular momenta, are also found.

1 Introduction

A very important development in the field of string theory has been achieved for the case of AdS/CFT duality [1] between strings and conformal field theories in various dimensions. The most developed case is the correspondence between strings living in $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM in four dimensions. Another example is the duality between strings on $AdS_4 \times CP^3$ background and $\mathcal{N} = 6$ super Chern-Simons-matter theory in three space-time dimensions. The main achievements in the above examples are due to the discovery of integrable structures on both sides of the correspondence. Many other cases have been considered also [2].

An interesting area of research is the AdS_3/CFT_2 duality [3]-[28], related to $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ string theory backgrounds where nontrivial two-form B field appears. For a review, see e.g [26].

The classical string solutions and their semiclassical limits, corresponding to large conserved charges [29], play important role in checking and understanding the AdS/CFT correspondence. Here we obtain a class of solutions corresponding to strings moving in the whole ten dimensional space-time $AdS_3 \times S^3 \times S^3 \times S^1$ with nonzero B -field.

The paper is organized as follows. In Sec.2 we describe the background. In Sec.3 we present our general approach to string dynamics in curved backgrounds with nonzero B -field. In Sec.4 we apply it to strings moving in $AdS_3 \times S^3 \times S^3 \times S^1$ with B -field. In Sec.5 we obtain the conserved charges for the case under consideration. Sec.6 is devoted to our concluding remarks.

2 The background

The metric of $AdS_3 \times S^3 \times S^3 \times S^1$ is

$$ds^2 = ds_{AdS_3}^2 + ds_{S_+^3}^2 + ds_{S_-^3}^2 + dw^2, \quad (2.1)$$

where w is the coordinate along S^1 . As found in [30], the radii of AdS_3 and of the two three-spheres satisfy the relation

$$\frac{1}{R_{AdS_3}^2} = \frac{1}{R_+^2} + \frac{1}{R_-^2}. \quad (2.2)$$

If we normalize the AdS_3 radius to one, (2.2) is solved by

$$\frac{1}{R_+^2} = \cos^2 \varphi, \quad \frac{1}{R_-^2} = \sin^2 \varphi. \quad (2.3)$$

According to [27], the metric on AdS_3 and the two three-spheres can be written as

$$ds_{AdS_3}^2 = - \left(\frac{1 + \frac{z_1^2 + z_2^2}{4}}{1 - \frac{z_1^2 + z_2^2}{4}} \right)^2 dt^2 + \left(\frac{1}{1 - \frac{z_1^2 + z_2^2}{4}} \right)^2 (dz_1^2 + dz_2^2), \quad (2.4)$$

$$ds_{S_+^3}^2 = \left(\frac{1 - \cos^2 \varphi \frac{y_3^2 + y_4^2}{4}}{1 + \cos^2 \varphi \frac{y_3^2 + y_4^2}{4}} \right)^2 d\phi_5^2 + \left(\frac{1}{1 + \cos^2 \varphi \frac{y_3^2 + y_4^2}{4}} \right)^2 (dy_3^2 + dy_4^2), \quad (2.5)$$

$$ds_{S_-^3}^2 = \left(\frac{1 - \sin^2 \varphi \frac{x_6^2 + x_7^2}{4}}{1 + \sin^2 \varphi \frac{x_6^2 + x_7^2}{4}} \right)^2 d\phi_8^2 + \left(\frac{1}{1 + \sin^2 \varphi \frac{x_6^2 + x_7^2}{4}} \right)^2 (dx_6^2 + dx_7^2). \quad (2.6)$$

The B -field in these coordinates is given by [27]

$$\begin{aligned} B = & \frac{q}{\left(1 - \frac{z_1^2 + z_2^2}{4}\right)^2} (z_1 dz_2 - z_2 dz_1) \wedge dt \\ & + \frac{q \cos \varphi}{\left(1 + \cos \varphi \frac{y_3^2 + y_4^2}{4}\right)^2} (y_3 dy_4 - y_4 dy_3) \wedge d\phi_5 \\ & + \frac{q \sin \varphi}{\left(1 + \sin \varphi \frac{x_6^2 + x_7^2}{4}\right)^2} (x_6 dx_7 - x_7 dx_6) \wedge d\phi_8, \end{aligned} \quad (2.7)$$

where the parameter q is related to the quantized coefficient k of the Wess-Zumino term by [27]

$$k = q\sqrt{\lambda}. \quad (2.8)$$

For our purposes here, we introduce new background coordinates:

$$z_1 = 2 \tanh \frac{\rho}{2} \cos \phi, \quad z_2 = 2 \tanh \frac{\rho}{2} \sin \phi, \quad (2.9)$$

$$\phi_5 = R_+ \phi_{2+} \quad (2.10)$$

$$\begin{aligned} y_3 &= R_+ w_1 = 2R_+ \tan \frac{\theta_+}{2} \cos \phi_{1+} \\ y_4 &= R_+ w_2 = 2R_+ \tan \frac{\theta_+}{2} \sin \phi_{1+}, \\ \phi_8 &= R_- \phi_{2-} \end{aligned} \quad (2.11)$$

$$\begin{aligned} x_6 &= R_- v_1 = 2R_- \tan \frac{\theta_-}{2} \cos \phi_{1-} \\ x_7 &= R_- v_2 = 2R_- \tan \frac{\theta_-}{2} \sin \phi_{1-}. \end{aligned}$$

As a consequence, the resulting description of the background becomes:

$$ds_{AdS_3}^2 = -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\phi^2, \quad (2.12)$$

or $(\sinh^2 \rho = r^2)$

$$ds_{AdS_3}^2 = -(1+r^2) \, dt^2 + (1+r^2)^{-1} dr^2 + r^2 d\phi^2 \quad (2.13)$$

$$\equiv g_{tt} \, dt^2 + g_{rr} dr^2 + g_{\phi\phi} d\phi^2$$

$$ds_{S_+^3}^2 = \frac{1}{\cos^2 \varphi} (d\theta_+^2 + \sin^2 \theta_+ d\phi_{1+}^2 + \cos^2 \theta_+ d\phi_{2+}^2) \quad (2.14)$$

$$\equiv g_{\theta_+\theta_+} d\theta_+^2 + g_{\phi_{1+}\phi_{1+}} d\phi_{1+}^2 + g_{\phi_{2+}\phi_{2+}} d\phi_{2+}^2,$$

$$ds_{S_-^3}^2 = \frac{1}{\sin^2 \varphi} (d\theta_-^2 + \sin^2 \theta_- d\phi_{1-}^2 + \cos^2 \theta_- d\phi_{2-}^2) \quad (2.15)$$

$$\equiv g_{\theta_-\theta_-} d\theta_-^2 + g_{\phi_{1-}\phi_{1-}} d\phi_{1-}^2 + g_{\phi_{2-}\phi_{2-}} d\phi_{2-}^2,$$

$$ds_{S^1}^2 = dw^2 \equiv g_{ww} dw^2,$$

$$B = qr^2 \, d\phi \wedge dt \quad (2.16)$$

$$+ \frac{q \sin^2 \theta_+}{\cos^2 \varphi \left(\cos^2 \frac{\theta_+}{2} + \frac{\sin^2 \frac{\theta_+}{2}}{\cos \varphi} \right)^2} d\phi_{1+} \wedge d\phi_{2+}$$

$$+ \frac{q \sin^2 \theta_-}{\sin^2 \varphi \left(\cos^2 \frac{\theta_-}{2} + \frac{\sin^2 \frac{\theta_-}{2}}{\sin \varphi} \right)^2} d\phi_{1-} \wedge d\phi_{2-}$$

$$\equiv b_{\phi t} \, d\phi \wedge dt + b_{\phi_{1+}\phi_{2+}} \, d\phi_{1+} \wedge d\phi_{2+} + b_{\phi_{1-}\phi_{2-}} \, d\phi_{1-} \wedge d\phi_{2-}.$$

3 The approach

Here, we will use the Polyakov type action for the bosonic string in a D -dimensional curved space-time with metric tensor $g_{MN}(x)$, interacting with a background 2-form gauge field $b_{MN}(x)$ via Wess-Zumino term

$$S^P = \int d^2 \xi \mathcal{L}^P, \quad \mathcal{L}^P = -\frac{1}{2} (T \sqrt{-\gamma} \gamma^{mn} G_{mn} - Q \varepsilon^{mn} B_{mn}),$$

$$\xi^m = (\xi^0, \xi^1) = (\tau, \sigma), \quad m, n = 0, 1,$$

where

$$G_{mn} = \partial_m X^M \partial_n X^N g_{MN}, \quad B_{mn} = \partial_m X^M \partial_n X^N b_{MN},$$

$$(\partial_m = \partial / \partial \xi^m, \quad M, N = 0, 1, \dots, D-1),$$

are the fields induced on the string worldsheet, γ is the determinant of the auxiliary worldsheet metric γ_{mn} , and γ^{mn} is its inverse. The position of the string in the background space-time is given by $x^M = X^M(\xi^m)$, and $T = 1/2\pi\alpha'$, Q are the string tension and charge, respectively. If we consider the action S^P as a bosonic part of a supersymmetric one, we have to put $Q = \pm T$. In what follows, $Q = T$.

The equations of motion for X^M following from S^P are:

$$\begin{aligned} & -g_{LK} [\partial_m (\sqrt{-\gamma} \gamma^{mn} \partial_n X^K) + \sqrt{-\gamma} \gamma^{mn} \Gamma_{MN}^K \partial_m X^M \partial_n X^N] \\ & = \frac{1}{2} H_{LMN} \epsilon^{mn} \partial_m X^M \partial_n X^N, \end{aligned} \quad (3.1)$$

where $(\partial_M = \partial/\partial x^M)$

$$\begin{aligned} \Gamma_{L,MN} &= g_{LK} \Gamma_{MN}^K = \frac{1}{2} (\partial_M g_{NL} + \partial_N g_{ML} - \partial_L g_{MN}), \\ H_{LMN} &= \partial_L b_{MN} + \partial_M b_{NL} + \partial_N b_{LM}, \end{aligned}$$

are the components of the symmetric connection corresponding to the metric g_{MN} , and the field strength of the gauge field b_{MN} respectively. The constraints are obtained by varying the action S^P with respect to γ_{mn} :

$$\delta_{\gamma_{mn}} S^P = 0 \Rightarrow (\gamma^{kl} \gamma^{mn} - 2\gamma^{km} \gamma^{ln}) G_{mn} = 0. \quad (3.2)$$

Further on, we will use *conformal gauge* $\gamma^{mn} = \eta^{mn} = \text{diag}(-1, 1)$ in which the string Lagrangian, the Virasoro constraints and the equations of motion take the following form:

$$\begin{aligned} \mathcal{L} &= \frac{T}{2} (G_{00} - G_{11} + 2B_{01}), \\ G_{00} + G_{11} &= 0, \quad G_{01} = 0, \\ g_{LK} [(\partial_0^2 - \partial_1^2) X^K + \Gamma_{MN}^K (\partial_0 X^M \partial_0 X^N - \partial_1 X^M \partial_1 X^N)] &= H_{LMN} \partial_0 X^M \partial_1 X^N. \end{aligned} \quad (3.3)$$

Now, we *suppose* that there exist some number of commuting Killing vector fields along part of X^M coordinates and split X^M into two parts

$$X^M = (X^\mu, X^a),$$

where X^μ are the isometric coordinates, while X^a are the non-isometric ones. The existence of isometric coordinates leads to the following conditions on the background fields:

$$\partial_\mu g_{MN} = 0, \quad \partial_\mu b_{MN} = 0. \quad (3.4)$$

Then from the string action, we can compute the conserved charges

$$Q_\mu = \int d\sigma \frac{\partial \mathcal{L}}{\partial(\partial_0 X^\mu)} \quad (3.5)$$

under the above conditions.

Next, we introduce the following ansatz for the string embedding

$$X^\mu(\tau, \sigma) = \Lambda^\mu \tau + \tilde{X}^\mu(\alpha\sigma + \beta\tau), \quad X^a(\tau, \sigma) = \tilde{X}^a(\alpha\sigma + \beta\tau), \quad (3.6)$$

where Λ^μ , α , β are arbitrary parameters. Further on, we will use the notation $\xi = \alpha\sigma + \beta\tau$.

Applying this ansatz, one can find that the equalities (3.3), (3.5) become

$$\mathcal{L} = \frac{T}{2} \left[-(\alpha^2 - \beta^2) g_{MN} \frac{d\tilde{X}^M}{d\xi} \frac{d\tilde{X}^N}{d\xi} + 2\Lambda^\mu (\beta g_{\mu N} + \alpha b_{\mu N}) \frac{d\tilde{X}^N}{d\xi} + \Lambda^\mu \Lambda^\nu g_{\mu\nu} \right], \quad (3.7)$$

$$G_{00} + G_{11} = (\alpha^2 + \beta^2) g_{MN} \frac{d\tilde{X}^M}{d\xi} \frac{d\tilde{X}^N}{d\xi} + 2\beta \Lambda^\mu g_{\mu N} \frac{d\tilde{X}^N}{d\xi} + \Lambda^\mu \Lambda^\nu g_{\mu\nu} = 0, \quad (3.8)$$

$$G_{01} = \alpha \beta g_{MN} \frac{d\tilde{X}^M}{d\xi} \frac{d\tilde{X}^N}{d\xi} + \alpha \Lambda^\mu g_{\mu N} \frac{d\tilde{X}^N}{d\xi} = 0, \quad (3.9)$$

$$\begin{aligned} & -(\alpha^2 - \beta^2) \left[g_{LK} \frac{d^2 \tilde{X}^K}{d\xi^2} + \Gamma_{L,MN} \frac{d\tilde{X}^M}{d\xi} \frac{d\tilde{X}^N}{d\xi} \right] + 2\beta \Lambda^\mu \Gamma_{L,\mu N} \frac{d\tilde{X}^N}{d\xi} + \Lambda^\mu \Lambda^\nu \Gamma_{L,\mu\nu} \\ & = \alpha \Lambda^\mu H_{L\mu N} \frac{d\tilde{X}^N}{d\xi}, \end{aligned} \quad (3.10)$$

$$Q_\mu = \frac{T}{\alpha} \int d\xi \left[(\beta g_{\mu N} + \alpha b_{\mu N}) \frac{d\tilde{X}^N}{d\xi} + \Lambda^\nu g_{\mu\nu} \right]. \quad (3.11)$$

Our next task is to try to solve the equations of motion (3.10) for the isometric coordinates, i.e. for $L = \lambda$. Due to the conditions (3.4) imposed on the background fields, we obtain that

$$\begin{aligned} \Gamma_{\lambda,ab} &= \frac{1}{2} (\partial_a g_{b\lambda} + \partial_b g_{a\lambda}), \quad \Gamma_{\lambda,\mu a} = \frac{1}{2} \partial_a g_{\mu\lambda} \quad \Gamma_{\lambda,\mu\nu} = 0, \\ H_{\lambda ab} &= \partial_a b_{b\lambda} + \partial_b b_{\lambda a}, \quad H_{\lambda\mu a} = \partial_a b_{\lambda\mu}, \quad H_{\lambda\mu\nu} = 0. \end{aligned}$$

By using this, one can find the following first integrals for \tilde{X}^μ :

$$\frac{d\tilde{X}^\mu}{d\xi} = \frac{1}{\alpha^2 - \beta^2} [g^{\mu\nu} (C_\nu - \alpha\Lambda^\rho b_{\nu\rho}) + \beta\Lambda^\mu] - g^{\mu\nu} g_{\nu a} \frac{d\tilde{X}^a}{d\xi}, \quad (3.12)$$

where C_ν are arbitrary integration constants. Therefore, according to our ansatz (3.6), the solutions for the string coordinates X^μ can be written as

$$X^\mu(\tau, \sigma) = \Lambda^\mu \tau + \frac{1}{\alpha^2 - \beta^2} \int d\xi [g^{\mu\nu} (C_\nu - \alpha\Lambda^\rho b_{\nu\rho}) + \beta\Lambda^\mu] - \int g^{\mu\nu} g_{\nu a} d\tilde{X}^a(\xi). \quad (3.13)$$

Now, let us turn to the remaining equations of motion corresponding to $L = a$, where

$$\begin{aligned} \Gamma_{a,\mu b} &= -\frac{1}{2}(\partial_a g_{b\mu} - \partial_b g_{a\mu}), & \Gamma_{a,\mu\nu} &= -\frac{1}{2}\partial_a g_{\mu\nu}, \\ H_{a\mu\nu} &= \partial_a b_{\mu\nu}, & H_{a\mu b} &= -\partial_a b_{b\mu} + \partial_b b_{a\mu}. \end{aligned}$$

Taking this into account and replacing the first integrals for \tilde{X}^μ already found, one can write these equations in the form (prime is used for $d/d\xi$)

$$(\alpha^2 - \beta^2) \left[h_{ab} \tilde{X}^{b''} + \Gamma_{a,bc}^h \tilde{X}^{b'} \tilde{X}^{c'} \right] = 2\partial_{[a} A_{b]} \tilde{X}^{b'} - \partial_a U, \quad (3.14)$$

where

$$h_{ab} = g_{ab} - g_{a\mu} g^{\mu\nu} g_{\nu b}, \quad \Gamma_{a,bc}^h = \frac{1}{2}(\partial_b h_{ca} + \partial_c h_{ba} - \partial_a h_{bc}) \quad (3.15)$$

$$A_a = g_{a\mu} g^{\mu\nu} (C_\nu - \alpha\Lambda^\rho b_{\nu\rho}) + \alpha\Lambda^\mu b_{a\mu}, \quad (3.16)$$

$$U = \frac{1/2}{\alpha^2 - \beta^2} [(C_\mu - \alpha\Lambda^\rho b_{\mu\rho}) g^{\mu\nu} (C_\nu - \alpha\Lambda^\lambda b_{\nu\lambda}) + \alpha^2 \Lambda^\mu \Lambda^\nu g_{\mu\nu}]. \quad (3.17)$$

The Virasoro constraints (3.8), (3.9) become:

$$\frac{1}{2}(\alpha^2 - \beta^2) h_{ab} \tilde{X}^{a'} \tilde{X}^{b'} + U = 0, \quad \alpha\Lambda^\mu C_\mu = 0. \quad (3.18)$$

Finally, let us write down the expressions for the conserved charges (3.11)

$$\begin{aligned} Q_\mu &= \frac{T}{\alpha^2 - \beta^2} \int d\xi \left[\frac{\beta}{\alpha} C_\mu + \alpha\Lambda^\nu g_{\mu\nu} + b_{\mu\nu} g^{\nu\rho} (C_\rho - \alpha\Lambda^\lambda b_{\rho\lambda}) \right. \\ &\quad \left. + (\alpha^2 - \beta^2) (b_{\mu a} - b_{\mu\nu} g^{\nu\rho} g_{\rho a}) \tilde{X}^{a'} \right]. \end{aligned} \quad (3.19)$$

4 String solutions in $AdS_3 \times S^3 \times S^3 \times S^1$ with B -field

In accordance with our notations

$$\begin{aligned}
\mu &= (t, \phi, \phi_{1+}, \phi_{2+}, \phi_{1-}, \phi_{2-}, w), \quad a = (r, \theta_+, \theta_-), \\
g_{\mu\nu} &= (g_{tt}, g_{\phi\phi}, g_{\phi_{1+}\phi_{1+}}, g_{\phi_{2+}\phi_{2+}}, g_{\phi_{1-}\phi_{1-}}, g_{\phi_{2-}\phi_{2-}}, g_{ww}), \\
g_{ab} &= (g_{rr}, g_{\theta_+\theta_+}, g_{\theta_-\theta_-}), \\
g_{a\mu} &= 0, \quad h_{ab} = g_{ab}, \\
b_{\mu\nu} &= (b_{\phi t}, b_{\phi_{1+}\phi_{2+}}, b_{\phi_{1-}\phi_{2-}}), \quad b_{a\nu} = 0, \\
A_a &= 0,
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
g_{tt} &= -(1 + r^2), \quad g_{rr} = (1 + r^2)^{-1}, \quad g_{\phi\phi} = r^2, \\
g_{\theta_+\theta_+} &= \frac{1}{\cos^2 \varphi}, \quad g_{\phi_{1+}\phi_{1+}} = \frac{1}{\cos^2 \varphi} \sin^2 \theta_+, \quad g_{\phi_{2+}\phi_{2+}} = \frac{1}{\cos^2 \varphi} \cos^2 \theta_+, \\
g_{\theta_-\theta_-} &= \frac{1}{\sin^2 \varphi}, \quad g_{\phi_{1-}\phi_{1-}} = \frac{1}{\sin^2 \varphi} \sin^2 \theta_-, \quad g_{\phi_{2-}\phi_{2-}} = \frac{1}{\sin^2 \varphi} \cos^2 \theta_-, \\
g_{ww} &= 1, \\
b_{t\phi} &= -qr^2, \\
b_{\phi_{1+}\phi_{2+}} &= \frac{q \sin^2 \theta_+}{\cos^2 \varphi \left(\cos^2 \frac{\theta_+}{2} + \frac{\sin^2 \frac{\theta_+}{2}}{\cos \varphi} \right)^2}, \\
b_{\phi_{1-}\phi_{2-}} &= \frac{q \sin^2 \theta_-}{\sin^2 \varphi \left(\cos^2 \frac{\theta_-}{2} + \frac{\sin^2 \frac{\theta_-}{2}}{\sin \varphi} \right)^2}.
\end{aligned} \tag{4.2}$$

The effective scalar potential (3.17) can be computed to be

$$U = \sum_{i=1}^4 U_i, \tag{4.3}$$

where

$$U_1(r) = \frac{1/2}{\alpha^2 - \beta^2} \left\{ -\frac{(C_t + \alpha\Lambda^\phi q r^2)^2}{1 + r^2} + \frac{(C_\phi - \alpha\Lambda^t q r^2)^2}{r^2} - \alpha^2 \left[(\Lambda^t)^2 (1 + r^2) - (\Lambda^\phi)^2 r^2 \right] \right\}, \quad (4.4)$$

$$U_2(\theta_+) = \frac{1/2}{\alpha^2 - \beta^2} \left\{ \left[C_{\phi_{1+}} - \frac{\alpha\Lambda^{\phi_{2+}} q \sin^2 \theta_+}{\cos^2 \varphi \left(\cos^2 \frac{\theta_+}{2} + \frac{\sin^2 \frac{\theta_+}{2}}{\cos \varphi} \right)^2} \right]^2 \frac{\cos^2 \varphi}{\sin^2 \theta_+} + \left[C_{\phi_{2+}} + \frac{\alpha\Lambda^{\phi_{1+}} q \sin^2 \theta_+}{\cos^2 \varphi \left(\cos^2 \frac{\theta_+}{2} + \frac{\sin^2 \frac{\theta_+}{2}}{\cos \varphi} \right)^2} \right]^2 \frac{\cos^2 \varphi}{\cos^2 \theta_+} + \frac{\alpha^2}{\cos^2 \varphi} \left[(\Lambda^{\phi_{1+}})^2 \sin^2 \theta_+ + (\Lambda^{\phi_{2+}})^2 \cos^2 \theta_+ \right] \right\}, \quad (4.5)$$

$$U_3(\theta_-) = \frac{1/2}{\alpha^2 - \beta^2} \left\{ \left[C_{\phi_{1-}} - \frac{\alpha\Lambda^{\phi_{2-}} q \sin^2 \theta_-}{\sin^2 \varphi \left(\cos^2 \frac{\theta_-}{2} + \frac{\sin^2 \frac{\theta_-}{2}}{\sin \varphi} \right)^2} \right]^2 \frac{\sin^2 \varphi}{\sin^2 \theta_-} + \left[C_{\phi_{2-}} + \frac{\alpha\Lambda^{\phi_{1-}} q \sin^2 \theta_-}{\sin^2 \varphi \left(\cos^2 \frac{\theta_-}{2} + \frac{\sin^2 \frac{\theta_-}{2}}{\sin \varphi} \right)^2} \right]^2 \frac{\sin^2 \varphi}{\cos^2 \theta_-} + \frac{\alpha^2}{\sin^2 \varphi} \left[(\Lambda^{\phi_{1-}})^2 \sin^2 \theta_- + (\Lambda^{\phi_{2-}})^2 \cos^2 \theta_- \right] \right\}, \quad (4.6)$$

$$U_4 = C_w^2 + (\Lambda^w)^2 = \text{const.} \quad (4.7)$$

By using that in the case under consideration the metric g_{ab} is diagonal, one can prove that the equations of motion (3.14) possess the following first integrals

$$\frac{d\tilde{X}^a}{d\xi} = \sqrt{\frac{C_a - 2U_a}{(\alpha^2 - \beta^2)g_{aa}}}, \quad (4.8)$$

where $\tilde{X}^a = (r, \theta_+, \theta_-)$, $C_a = (C_r, C_{\theta_+}, C_{\theta_-})$ are arbitrary integration constants and $U_a = (U_1(r), U_2(\theta_+), U_3(\theta_-))$.

The replacement of (4.8) into (3.18) reduces the first Virasoro constraint to the following equality

$$C_r + C_{\theta_+} + C_{\theta_-} = 0. \quad (4.9)$$

Thus, the Virasoro constraints are simplified to relations between the integration constants and embedding parameters on this type of string solutions.

4.1 Solutions in AdS_3

For the AdS_3 subspace, (4.8) gives

$$d\xi = \frac{dr}{\sqrt{\frac{(C_r - 2U_1(r))(1+r^2)}{\alpha^2 - \beta^2}}}.$$

By using the expression (4.4) for $U_1(r)$ and introducing the variable $y = r^2$, one obtains

$$d\xi = \frac{\alpha^2 - \beta^2}{2\alpha\sqrt{(1-q^2)\left[(\Lambda^\phi)^2 - (\Lambda^t)^2\right]}} \frac{dy}{\sqrt{(y_p - y)(y - y_m)(y - y_n)}}, \quad (4.10)$$

where

$$y_p > y > y_m \geq 0, \quad y_n < 0,$$

and y_p, y_m, y_n satisfy the equalities

$$\begin{aligned} y_p + y_m + y_n &= \frac{1}{\alpha^2(1-q^2)\left[(\Lambda^\phi)^2 - (\Lambda^t)^2\right]} \\ &\left[C_r(\alpha^2 - \beta^2) - \alpha\left(\alpha(\Lambda^\phi)^2 - 2\alpha(\Lambda^t)^2 - 2q(C_\phi\Lambda^t + C_t\Lambda^\phi) + q^2\alpha(\Lambda^t)^2\right)\right], \\ y_py_m + y_py_n + y_my_n &= -\frac{1}{\alpha^2(1-q^2)\left[(\Lambda^\phi)^2 - (\Lambda^t)^2\right]} \\ &\left[C_r(\alpha^2 - \beta^2) + C_t^2 - C_\phi^2 + \alpha^2(\Lambda^t)^2 + 2q\alpha C_\phi\Lambda^t\right], \\ y_py_my_n &= -\frac{C_\phi^2}{\alpha^2(1-q^2)\left[(\Lambda^\phi)^2 - (\Lambda^t)^2\right]}. \end{aligned} \quad (4.11)$$

Integrating (4.10) and inverting

$$\xi(y) = \frac{\alpha^2 - \beta^2}{\alpha \sqrt{(1 - q^2) [(\Lambda^\phi)^2 - (\Lambda^t)^2]} (y_p - y_n)} \mathbf{F} \left(\arcsin \sqrt{\frac{y_p - y}{y_p - y_m}}, \frac{y_p - y_m}{y_p - y_n} \right)$$

to $y(\xi)$, one finds the following solution

$$y(\xi) = (y_p - y_n) \mathbf{DN}^2 \left[\frac{\alpha \sqrt{(1 - q^2) [(\Lambda^\phi)^2 - (\Lambda^t)^2]} (y_p - y_n)}{\alpha^2 - \beta^2} \xi, \frac{y_p - y_m}{y_p - y_n} \right] + y_n, \quad (4.12)$$

where \mathbf{F} is the incomplete elliptic integral of first kind and \mathbf{DN} is one of the Jacobi elliptic functions.

Now we are going to find the solutions for the isometric coordinates $t(\xi)$ and $\phi(\xi)$. In accordance with (3.12), the first integrals for \tilde{X}^t and \tilde{X}^ϕ can be computed to be given by

$$\begin{aligned} \frac{d\tilde{X}^t}{d\xi} &= \frac{1}{\alpha^2 - \beta^2} \left[\beta \Lambda^t - q \alpha \Lambda^\phi - \frac{C_t - q \alpha \Lambda^\phi}{1 + y} \right], \\ \frac{d\tilde{X}^\phi}{d\xi} &= \frac{1}{\alpha^2 - \beta^2} \left(\beta \Lambda^\phi - q \alpha \Lambda^t + \frac{C_\phi}{y} \right). \end{aligned}$$

Integrating and using (4.10), we obtain

$$t(\tau, \xi) = \Lambda^t \tau + \frac{1}{\alpha \sqrt{(1 - q^2) [(\Lambda^\phi)^2 - (\Lambda^t)^2]} (y_p - y_n)} \quad (4.13)$$

$$\begin{aligned} & \left[(\beta \Lambda^t - q \alpha \Lambda^\phi) \mathbf{F} \left(\arcsin \sqrt{\frac{y_p - y}{y_p - y_m}}, \frac{y_p - y_m}{y_p - y_n} \right) \right. \\ & \left. - \frac{C_t - q \alpha \Lambda^\phi}{1 + y_p} \mathbf{\Pi} \left(\arcsin \sqrt{\frac{y_p - y}{y_p - y_m}}, \frac{y_p - y_m}{y_p - y_n}, \frac{y_p - y_m}{y_p - y_n} \right) \right], \end{aligned}$$

$$\phi(\tau, \xi) = \Lambda^\phi \tau + \frac{1}{\alpha \sqrt{(1 - q^2) [(\Lambda^\phi)^2 - (\Lambda^t)^2]} (y_p - y_n)} \quad (4.14)$$

$$\begin{aligned} & \left[(\beta \Lambda^\phi - q \alpha \Lambda^t) \mathbf{F} \left(\arcsin \sqrt{\frac{y_p - y}{y_p - y_m}}, \frac{y_p - y_m}{y_p - y_n} \right) \right. \\ & \left. + \frac{C_\phi}{y_p} \mathbf{\Pi} \left(\arcsin \sqrt{\frac{y_p - y}{y_p - y_m}}, \frac{y_p - y_m}{y_p}, \frac{y_p - y_m}{y_p - y_n} \right) \right], \end{aligned}$$

where $\mathbf{\Pi}$ is the incomplete elliptic integral of third kind.

4.2 Solutions on the two S^3

It is clear from (2.14) and (2.15) that the solutions on the two three-spheres S_+^3 and S_-^3 can be obtained from each other by the exchanges $\sin \varphi \leftrightarrow \cos \varphi$ and $+ \leftrightarrow -$ in the subscripts of the coordinates, the corresponding integration constants, and in the superscripts of the embedding parameters. That is why we are going to present here the string solutions for one of the spheres only, say S_-^3 .

The non-isometric coordinate on S_-^3 is θ_- for which the first integral (4.8) reads

$$\frac{d\theta_-}{d\xi} = \sqrt{\frac{\sin^2 \varphi}{\alpha^2 - \beta^2} [C_{\theta_-} - 2U_3(\theta_-)]}. \quad (4.15)$$

$U_3(\theta_-)$ is given in (4.6).

Now we introduce the variable

$$\gamma = \cos^2 \frac{\theta_-}{2}.$$

This allows us to rewrite (4.15) in the following form

$$\begin{aligned} d\xi = & \frac{\alpha^2 - \beta^2}{\sin \varphi} (-a_{10})^{-1/2} (1 - (1 - \sin \varphi)\gamma)^2 (2\gamma - 1) \\ & [(\gamma_1 - \gamma)(\gamma - \gamma_2)(\gamma - \gamma_3)(\gamma - \gamma_4)(\gamma - \gamma_5)(\gamma - \gamma_6) \\ & (\gamma - \gamma_7)(\gamma - \gamma_8)(\gamma - \gamma_9)(\gamma - \gamma_{10})]^{-1/2} d\gamma, \quad \gamma_1 = \gamma_{max}. \end{aligned} \quad (4.16)$$

The corresponding computations are given in an Appendix. Next, we integrate

$$\begin{aligned} \xi = & \frac{\alpha^2 - \beta^2}{\sin \varphi} (-a_{10})^{-1/2} \int_{\gamma}^{\gamma_{max}} (1 - (1 - \sin \varphi)u)^2 (2u - 1) \\ & [(\gamma_{max} - u)(u - \gamma_2)(u - \gamma_3)(u - \gamma_4)(u - \gamma_5)(u - \gamma_6) \\ & (u - \gamma_7)(u - \gamma_8)(u - \gamma_9)(u - \gamma_{10})]^{-1/2} du, \end{aligned} \quad (4.17)$$

and introduce new integration variable

$$\delta = \frac{\gamma_{max} - u}{\gamma_{max} - \gamma}.$$

Then (4.17) becomes

$$\begin{aligned} \xi(\gamma) = & \frac{\alpha^2 - \beta^2}{\sin \varphi} (-a_{10})^{-1/2} (1 - \gamma_{max}(1 - \sin \varphi))^2 (2\gamma_{max} - 1)(\gamma_{max} - \gamma)^{1/2} \quad (4.18) \\ & \left[\prod_{i=2}^{10} (\gamma_{max} - \gamma_i) \right]^{-1/2} \int_0^1 \delta^{-1/2} \left(1 + \frac{(\gamma_{max} - \gamma)(1 - \sin \varphi)}{(1 - \gamma_{max}(1 - \sin \varphi))} \delta \right)^2 \\ & \left(1 - \frac{2(\gamma_{max} - \gamma)}{2\gamma_{max} - 1} \delta \right) \prod_{i=2}^{10} \left(1 - \frac{\gamma_{max} - \gamma}{\gamma_{max} - \gamma_i} \delta \right)^{-1/2} d\delta. \end{aligned}$$

Comparing the integral in (4.18) with the integral representation of the Lauricella hypergeometric functions of many variables $F_D^{(n)}$ [31]

$$\begin{aligned} F_D^{(n)}(a; b_1, \dots, b_n; c; z_1, \dots, z_n) = & \quad (4.19) \\ & \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \delta^{a-1} (1-\delta)^{c-a-1} (1-z_1\delta)^{-b_1} \dots (1-z_n\delta)^{-b_n} d\delta, \\ & Re(a) > 0, \quad Re(c-a) > 0, \end{aligned}$$

one finds

$$\begin{aligned} \xi(\gamma) = & 2 \frac{\alpha^2 - \beta^2}{\sin \varphi} (-a_{10})^{-1/2} (1 - \gamma_{max}(1 - \sin \varphi))^2 (2\gamma_{max} - 1)(\gamma_{max} - \gamma)^{1/2} \quad (4.20) \\ & \left[\prod_{i=2}^{10} (\gamma_{max} - \gamma_i) \right]^{-1/2} F_D^{(11)}(1/2; b_1, \dots, b_{11}; 3/2; z_1, \dots, z_{11}), \end{aligned}$$

where

$$\begin{aligned} b_1 = -2, \quad z_1 = & -\frac{(\gamma_{max} - \gamma)(1 - \sin \varphi)}{1 - \gamma_{max}(1 - \sin \varphi)}, \quad (4.21) \\ b_2 = -1, \quad z_2 = & \frac{2(\gamma_{max} - \gamma)}{2\gamma_{max} - 1}, \\ b_k = 1/2, \quad z_k = & \frac{\gamma_{max} - \gamma}{\gamma_{max} - \gamma_{k-1}}, \quad k = 3, \dots, 11. \end{aligned}$$

This is our final result for $\xi(\gamma)$. Unfortunately this solution is not invertible, so we can not write down $\gamma(\xi)$.

Now, let us proceed with obtaining the solutions for the isometric coordinates on S_-^3 . From (3.12) one finds the following first integrals:

$$\frac{d\tilde{X}^{\phi_{1-}}}{d\xi} = \frac{1}{\alpha^2 - \beta^2} \left\{ \beta \Lambda^{\phi_{1-}} + \sin^2 \varphi \left[\frac{C_{\phi_{1-}}}{\sin^2 \theta_-} - \frac{q\alpha \Lambda^{\phi_{2-}}}{\sin^2 \varphi \left(\cos^2 \frac{\theta_-}{2} + \frac{\sin^2 \frac{\theta_-}{2}}{\sin \varphi} \right)^2} \right] \right\}, \quad (4.22)$$

$$\frac{d\tilde{X}^{\phi_{2-}}}{d\xi} = \frac{1}{\alpha^2 - \beta^2} \left\{ \beta \Lambda^{\phi_{2-}} + \frac{\sin^2 \varphi}{\cos^2 \theta_-} \left[C_{\phi_{2-}} + \frac{q\alpha \Lambda^{\phi_{1-}} \sin^2 \theta_-}{\sin^2 \varphi \left(\cos^2 \frac{\theta_-}{2} + \frac{\sin^2 \frac{\theta_-}{2}}{\sin \varphi} \right)^2} \right] \right\}. \quad (4.23)$$

Introducing the variable γ and using (4.16), one arrives at

$$\begin{aligned} \tilde{X}^{\phi_{1-}} \equiv \tilde{\phi}_{1-} &= 2 \sin \varphi (-a_{10})^{-1/2} (1 - \gamma_{\max} (1 - \sin \varphi))^2 (2\gamma_{\max} - 1) (\gamma_{\max} - \gamma)^{1/2} \quad (4.24) \\ &\quad \left[\prod_{i=2}^{10} (\gamma_{\max} - \gamma_i) \right]^{-1/2} \left[\frac{\beta \Lambda^{\phi_{1-}}}{\sin^2 \varphi} F_D^{(11)}(1/2; b_1, \dots, b_{11}; 3/2; z_1, \dots, z_{11}) \right. \\ &\quad + \frac{C_{\phi_{1-}}}{4(1 - \gamma_{\max})\gamma_{\max}} F_D^{(13)}(1/2; b_1, \dots, b_{11}, b_{12}, b_{13}; 3/2; z_1, \dots, z_{11}, z_{12}, z_{13}) \\ &\quad \left. - q\alpha \Lambda^{\phi_{2-}} (1 - \gamma_{\max} (1 - \sin \varphi))^{-2} F_D^{(10)}(1/2; b_2, \dots, b_{11}; 3/2; z_2, \dots, z_{11}) \right], \end{aligned}$$

$$\begin{aligned} \tilde{X}^{\phi_{2-}} \equiv \tilde{\phi}_{2-} &= 2 \sin \varphi (-a_{10})^{-1/2} (1 - \gamma_{\max} (1 - \sin \varphi))^2 (2\gamma_{\max} - 1) (\gamma_{\max} - \gamma)^{1/2} \quad (4.25) \\ &\quad \left[\prod_{i=2}^{10} (\gamma_{\max} - \gamma_i) \right]^{-1/2} \left[\frac{\beta \Lambda^{\phi_{2-}}}{\sin^2 \varphi} F_D^{(11)}(1/2; b_1, \dots, b_{11}; 3/2; z_1, \dots, z_{11}) \right. \\ &\quad + \frac{C_{\phi_{2-}}}{(2\gamma_{\max} - 1)^2} F_D^{(12)}(1/2; b_1, \dots, b_{11}, c_{12}; 3/2; z_1, \dots, z_{11}, y_{12}) \\ &\quad + 4q\alpha \Lambda^{\phi_{1-}} (2\gamma_{\max} - 1)^{-2} \gamma_{\max} (1 - \gamma_{\max}) (1 - \gamma_{\max} (1 - \sin \varphi))^{-2} \times \\ &\quad \left. F_D^{(15)}(1/2; b_1, \dots, b_{11}, c_{12}, c_{13}, c_{14}, c_{15}; 3/2; z_1, \dots, z_{11}, y_{12}, z_{12}, z_{13}, y_{15}) \right], \end{aligned}$$

where

$$\begin{aligned} b_{12} &= 1, & z_{12} &= -\frac{\gamma_{\max} - \gamma}{1 - \gamma_{\max}}, \\ b_{13} &= 1, & z_{13} &= \frac{\gamma_{\max} - \gamma}{\gamma_{\max}}, \\ c_{12} &= 2, & y_{12} &= 2 \frac{\gamma_{\max} - \gamma}{2\gamma_{\max} - 1}, \\ c_{13} &= -1, & c_{14} &= -1, \\ c_{15} &= 2, & y_{15} &= -\frac{(\gamma_{\max} - \gamma)(1 - \sin \varphi)}{1 - \gamma_{\max}(1 - \sin \varphi)}. \end{aligned} \quad (4.26)$$

Therefore, according to (3.13), the solutions for the isometric coordinates on S_-^3 are given by

$$\phi_{1-}(\tau, \xi) = \Lambda^{\phi_{1-}} \tau + \tilde{\phi}_{1-}(\xi), \quad \phi_{2-}(\tau, \xi) = \Lambda^{\phi_{2-}} \tau + \tilde{\phi}_{2-}(\xi).$$

5 Conserved charges

The expressions for the conserved charges corresponding to the isometric coordinates can be found from (3.19) to be

$$Q_t \equiv -E_s = \frac{T}{\alpha^2 - \beta^2} \int \left[\frac{\beta}{\alpha} C_t - \alpha \Lambda^t (1 + r^2) - q (C_\phi - q \alpha \Lambda^t r^2) \right] d\xi, \quad (5.1)$$

$$Q_\phi \equiv S = \frac{T}{\alpha^2 - \beta^2} \int \left[\frac{\beta}{\alpha} C_\phi - \alpha \Lambda^\phi r^2 - q \frac{r^2}{1 + r^2} (C_t + q \alpha \Lambda^\phi r^2) \right] d\xi, \quad (5.2)$$

$$Q_{\phi_{1+}} \equiv J_{1+} = \frac{T}{\alpha^2 - \beta^2} \int \left[\frac{\beta}{\alpha} C_{\phi_{1+}} + \frac{\alpha \Lambda^{\phi_{1+}}}{\cos^2 \varphi} \sin^2 \theta_+ \right. \\ \left. + \frac{q \sin^2 \theta_+}{\cos^2 \theta_+ \left(\cos^2 \frac{\theta_+}{2} + \frac{\sin^2 \frac{\theta_+}{2}}{\cos \varphi} \right)^2} \left(C_{\phi_{2+}} + \frac{q \alpha \Lambda^{\phi_{1+}} \sin^2 \theta_+}{\cos^2 \varphi \left(\cos^2 \frac{\theta_+}{2} + \frac{\sin^2 \frac{\theta_+}{2}}{\cos \varphi} \right)^2} \right) \right] d\xi, \quad (5.3)$$

$$Q_{\phi_{2+}} \equiv J_{2+} = \frac{T}{\alpha^2 - \beta^2} \int \left[\frac{\beta}{\alpha} C_{\phi_{2+}} + \frac{\alpha \Lambda^{\phi_{2+}}}{\cos^2 \varphi} \cos^2 \theta_+ \right. \\ \left. - \frac{q}{\left(\cos^2 \frac{\theta_+}{2} + \frac{\sin^2 \frac{\theta_+}{2}}{\cos \varphi} \right)^2} \left(C_{\phi_{1+}} - \frac{q \alpha \Lambda^{\phi_{2+}} \sin^2 \theta_+}{\cos^2 \varphi \left(\cos^2 \frac{\theta_+}{2} + \frac{\sin^2 \frac{\theta_+}{2}}{\cos \varphi} \right)^2} \right) \right] d\xi, \quad (5.4)$$

$$Q_{\phi_{1-}} \equiv J_{1-} = \frac{T}{\alpha^2 - \beta^2} \int \left[\frac{\beta}{\alpha} C_{\phi_{1-}} + \frac{\alpha \Lambda^{\phi_{1-}}}{\sin^2 \varphi} \sin^2 \theta_- \right. \\ \left. + \frac{q \sin^2 \theta_-}{\cos^2 \theta_- \left(\cos^2 \frac{\theta_-}{2} + \frac{\sin^2 \frac{\theta_-}{2}}{\sin \varphi} \right)^2} \left(C_{\phi_{2-}} + \frac{q \alpha \Lambda^{\phi_{1-}} \sin^2 \theta_-}{\sin^2 \varphi \left(\cos^2 \frac{\theta_-}{2} + \frac{\sin^2 \frac{\theta_-}{2}}{\sin \varphi} \right)^2} \right) \right] d\xi, \quad (5.5)$$

$$Q_{\phi_{2-}} \equiv J_{2-} = \frac{T}{\alpha^2 - \beta^2} \int \left[\frac{\beta}{\alpha} C_{\phi_{2-}} + \frac{\alpha \Lambda^{\phi_{2-}}}{\sin^2 \varphi} \cos^2 \theta_- \right. \\ \left. - \frac{q}{\left(\cos^2 \frac{\theta_-}{2} + \frac{\sin^2 \frac{\theta_-}{2}}{\sin \varphi} \right)^2} \left(C_{\phi_{1-}} - \frac{q \alpha \Lambda^{\phi_{2-}} \sin^2 \theta_-}{\sin^2 \varphi \left(\cos^2 \frac{\theta_-}{2} + \frac{\sin^2 \frac{\theta_-}{2}}{\sin \varphi} \right)^2} \right) \right] d\xi. \quad (5.6)$$

Here we introduced the following notations: E_s is the string energy, S is the spin of the string in AdS_3 , J_{1+} , J_{2+} , J_{1-} , J_{2-} , are the angular momenta on the two three spheres S_{\pm}^3 .

In order to compute E_s and S , we introduce the variable $y = r^2$ and use the expression (4.10) for $d\xi$ in AdS_3 . This leads to the following results

$$E_s = \frac{2T}{\sqrt{(1-q^2) \left[(\Lambda^\phi)^2 - (\Lambda^t)^2 \right] (y_p - y_n)}} \quad (5.7)$$

$$\left[\left(\Lambda^t - \frac{\beta}{\alpha^2} C_t + q \frac{C_\phi}{\alpha} \right) \mathbf{K} \left(1 - \frac{y_m - y_n}{y_p - y_n} \right) + \right. \\ \left. (1 - q^2) \Lambda^t \left(y_n \mathbf{K} \left(1 - \frac{y_m - y_n}{y_p - y_n} \right) + (y_p - y_n) \mathbf{E} \left(1 - \frac{y_m - y_n}{y_p - y_n} \right) \right) \right],$$

$$S = \frac{2T}{\sqrt{(1-q^2) \left[(\Lambda^\phi)^2 - (\Lambda^t)^2 \right] (y_p - y_n)}} \quad (5.8)$$

$$\left[\left(\frac{\beta}{\alpha^2} C_\phi - q \frac{C_t}{\alpha} + \Lambda^\phi q^2 \right) \mathbf{K} \left(1 - \frac{y_m - y_n}{y_p - y_n} \right) + \right. \\ \left. (1 - q^2) \Lambda^\phi \left(y_n \mathbf{K} \left(1 - \frac{y_m - y_n}{y_p - y_n} \right) + (y_p - y_n) \mathbf{E} \left(1 - \frac{y_m - y_n}{y_p - y_n} \right) \right) \right. \\ \left. + \frac{q \frac{C_t}{\alpha} - q^2 \Lambda^\phi}{1 + y_p} \mathbf{\Pi} \left(\frac{y_p - y_m}{1 + y_p}, 1 - \frac{y_m - y_n}{y_p - y_n} \right) \right],$$

where \mathbf{K} , \mathbf{E} and $\mathbf{\Pi}$ are the complete elliptic integrals of first, second and third kind.

To find the final expressions for the angular momenta, we now introduce the variable γ ($\gamma = \cos^2 \frac{\theta_+}{2}$ for S_+^3 and $\gamma = \cos^2 \frac{\theta_-}{2}$ for S_-^3). Here we present the results for J_{1-} and J_{2-}

only. By using (4.16) we derive:

$$\begin{aligned}
J_{1-} = & \frac{2\pi T}{\sin \varphi} (-a_{10})^{-1/2} (1 - \gamma_{max}(1 - \sin \varphi))^2 (2\gamma_{max} - 1)(\gamma_{max} - \gamma_{min})^{1/2} \\
& \left[\prod_{i=2}^{10} (\gamma_{max} - \gamma_i) \right]^{-1/2} \times \\
& \left[\frac{\beta}{\alpha} C_{\phi_{1-}} F_D^{(10)}(1/2; b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_{11}; 1; Z_1, \dots, Z_{k-1}, Z_{k+1}, \dots, Z_{11}) \right. \\
& + \frac{4\alpha\Lambda^{\phi_{1-}}}{\sin^2 \varphi} \gamma_{max}(1 - \gamma_{max}) \times \\
& F_D^{(12)}(1/2; b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_{11}, b_{12}, b_{13}; 1; Z_1, \dots, Z_{k-1}, Z_{k+1}, \dots, Z_{11}, Z_{12}, Z_{13}) \\
& + 4q \sin^2 \varphi \gamma_{max}(1 - \gamma_{max}) \left(\frac{C_{\phi_{2-}}}{(2\gamma_{max} - 1)^2 (1 - \gamma_{max}(1 - \sin \varphi))^2} \times \right. \\
& F_D^{(11)}(1/2; b_2, \dots, b_{k-1}, b_{k+1}, \dots, b_{11}, b_{12}, b_{13}; 1; Z_2, \dots, Z_{k-1}, Z_{k+1}, \dots, Z_{11}, Z_{12}, Z_{13}) \\
& + \frac{4q\alpha\Lambda^{\phi_{1-}} \gamma_{max}(1 - \gamma_{max})}{(2\gamma_{max} - 1)^2 (1 - \gamma_{max}(1 - \sin \varphi))^4} \times \\
& \left. \left. F_D^{(12)}(1/2; 2, 1, b_3, \dots, b_{k-1}, b_{k+1}, \dots, b_{11}, -2, -2; 1; Z_1, \dots, Z_{k-1}, Z_{k+1}, \dots, Z_{11}, Z_{12}, Z_{13}) \right) \right] ,
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
J_{2-} = & \frac{2\pi T}{\sin \varphi} (-a_{10})^{-1/2} (1 - \gamma_{max}(1 - \sin \varphi))^2 (2\gamma_{max} - 1)(\gamma_{max} - \gamma_{min})^{1/2} \\
& \left[\prod_{i=2}^{10} (\gamma_{max} - \gamma_i) \right]^{-1/2} \times \\
& \left[\frac{\beta}{\alpha} C_{\phi_{2-}} F_D^{(10)}(1/2; b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_{11}; 1; Z_1, \dots, Z_{k-1}, Z_{k+1}, \dots, Z_{11}) \right. \\
& + \frac{\alpha\Lambda^{\phi_{2-}}}{\sin^2 \varphi} (2\gamma_{max} - 1)^2 \times \\
& F_D^{(10)}(1/2; b_1, -3, b_3, \dots, b_{k-1}, b_{k+1}, \dots, b_{11}; 1; Z_1, \dots, Z_{k-1}, Z_{k+1}, \dots, Z_{11}) \\
& - q \sin^2 \varphi C_{\phi_{1-}} (1 - \gamma_{max}(1 - \sin \varphi)) \times \\
& F_D^{(9)}(1/2; b_2, \dots, b_{k-1}, b_{k+1}, \dots, b_{11}; 1; Z_2, \dots, Z_{k-1}, Z_{k+1}, \dots, Z_{11}) \\
& + 4q^2 \alpha \Lambda^{\phi_{2-}} \sin^2 \varphi \gamma_{max}(1 - \gamma_{max})(1 - \gamma_{max}(1 - \sin \varphi))^{-4} \times \\
& \left. F_D^{(13)}(1/2; 2, -1, b_3, \dots, b_{k-1}, b_{k+1}, \dots, b_{11}, -1, -1; 1; Z_1, \dots, Z_{k-1}, Z_{k+1}, \dots, Z_{11}, Z_{12}, Z_{13}) \right] ,
\end{aligned} \tag{5.10}$$

where Z_k are related to the previous z_k by the change $\gamma \rightarrow \gamma_{min}$.

In writing (5.9), (5.10), we used the following property of the hypergeometric functions

$F_D^{(n)}$:

$$F_D^{(n)}(a; b_1, \dots, b_n; c; z_1, \dots, z_{k-1}, 1, z_{k+1}, \dots, z_n) = \frac{\Gamma(c)\Gamma(c-a-b_k)}{\Gamma(c-a)\Gamma(c-b_k)} \times \quad (5.11)$$

$$F_D^{(n-1)}(a; b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_n; c-b_k; z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n).$$

It follows from the integral representation (4.19). We needed to use this property in order to take into account that for some k

$$Z_k = \frac{\gamma_{max} - \gamma_{min}}{\gamma_{max} - \gamma_{k-1}} = 1,$$

i.e. $\gamma_{k-1} = \gamma_{min} \geq 0$.

6 Concluding remarks

Here we considered strings living in $AdS_3 \times S^3 \times S^3 \times S^1$ with nonzero 2-form B -field. By using specific ansatz for the string embedding, we obtained a class of solutions corresponding to strings moving in the whole ten dimensional space-time. For the AdS_3 subspace, these solutions are given in terms of incomplete elliptic integrals as expected. For the two three-spheres, they are expressed in terms of Lauricella hypergeometric functions of many variables. The same is true for the corresponding conserved angular momenta related to the isometries of the three-spheres. This is in contrast with the case of $AdS_3 \times S^3 \times T^4$ background with B -field, where the solutions for the string coordinates on S^3 are given in terms of incomplete elliptic integrals [25, 28]. The complications here arise because of the specific form of the B -field for this supergravity solution (see (2.16)).

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Appendix

Here we explain how (4.16) is derived. First we represent $d\xi$ as

$$d\xi = \frac{\alpha^2 - \beta^2}{\sin \varphi} (1 - (1 - \sin \varphi)\gamma)^2 (2\gamma - 1) \sum_{i=0}^{10} a_i \gamma^i d\gamma,$$

where

$$a_0 = -\frac{1}{4}C_{\phi_{1-}}^2 \sin^2 \varphi,$$

$$a_1 = C_{\theta_-}(\alpha^2 - \beta^2) - \alpha^2 (\Lambda^{\phi_{2-}})^2 \csc^2 \varphi \\ - \sin^2 \varphi \left[(C_{\phi_{1-}})^2 \sin \varphi + (C_{\phi_{2-}})^2 - 2C_{\phi_{1-}} (C_{\phi_{1-}} + q\alpha\Lambda^{\phi_{2-}}) \right],$$

$$a_2 = -9C_{\theta_-}(\alpha^2 - \beta^2) - 4\alpha^2 (\Lambda^{\phi_{2-}})^2 \csc \varphi + \alpha^2 \left[13 (\Lambda^{\phi_{2-}})^2 - 4 (\Lambda^{\phi_{1-}})^2 \right] \csc^2 \varphi \\ + \frac{1}{2} \sin \varphi \left(8C_{\theta_-}(\alpha^2 - \beta^2) - \sin \varphi \left(13 (C_{\phi_{1-}})^2 - 2C_{\phi_{2-}} (5C_{\phi_{2-}} - 8q\alpha\Lambda^{\phi_{1-}}) \right. \right. \\ \left. \left. + 28q\alpha C_{\phi_{1-}} \Lambda^{\phi_{2-}} + 8q^2\alpha^2 (\Lambda^{\phi_{2-}})^2 + \sin \varphi \left(8 (C_{\phi_{2-}})^2 \right. \right. \right. \\ \left. \left. \left. - 2C_{\phi_{1-}} (7C_{\phi_{1-}} + 4q\alpha\Lambda^{\phi_{2-}}) + 3 (C_{\phi_{1-}})^2 \sin \varphi \right) \right) \right),$$

$$a_3 = 34C_{\theta_-}(\alpha^2 - \beta^2) - 6\alpha^2 (\Lambda^{\phi_{2-}})^2 - 16\alpha^2 \left((\Lambda^{\phi_{1-}})^2 - 3 (\Lambda^{\phi_{2-}})^2 \right) \csc \varphi \\ + 2\alpha^2 \left(20 (\Lambda^{\phi_{1-}})^2 - 37 (\Lambda^{\phi_{2-}})^2 \right) \csc^2 \varphi + \sin \varphi \times \\ \left(-32C_{\theta_-}(\alpha^2 - \beta^2) + \sin \varphi \left(11 (C_{\phi_{1-}})^2 + 38q\alpha C_{\phi_{1-}} \Lambda^{\phi_{2-}} \right. \right. \\ \left. \left. + 2 \left(3C_{\theta_-}(\alpha^2 - \beta^2) - 5 (C_{\phi_{2-}})^2 + 16q\alpha C_{\phi_{2-}} \Lambda^{\phi_{1-}} \right. \right. \right. \\ \left. \left. \left. - 4q^2\alpha^2 \left(2 (\Lambda^{\phi_{1-}})^2 - 3 (\Lambda^{\phi_{2-}})^2 \right) \right) \right) - \sin \varphi \left(19 (C_{\phi_{1-}})^2 \right. \right. \\ \left. \left. - 16C_{\phi_{2-}} (C_{\phi_{2-}} - q\alpha\Lambda^{\phi_{1-}}) + 24q\alpha C_{\phi_{1-}} \Lambda^{\phi_{2-}} + \sin \varphi \times \right. \right. \\ \left. \left. \left(6 (C_{\phi_{2-}})^2 - 9 (C_{\phi_{1-}})^2 + (C_{\phi_{1-}})^2 \sin \varphi - 2q\alpha C_{\phi_{1-}} \Lambda^{\phi_{2-}} \right) \right) \right),$$

$$a_4 = -70C_{\theta_-}(\alpha^2 - \beta^2) - 6\alpha^2 \left(4 (\Lambda^{\phi_{1-}})^2 - 11 (\Lambda^{\phi_{2-}})^2 \right) \\ + 8\alpha^2 \left(18 (\Lambda^{\phi_{1-}})^2 - 31 (\Lambda^{\phi_{2-}})^2 \right) \csc \varphi - 2\alpha^2 \left(86 (\Lambda^{\phi_{1-}})^2 - 121 (\Lambda^{\phi_{2-}})^2 \right) \csc^2 \varphi \\ + \frac{1}{4} \sin \varphi \left(416C_{\theta_-}(\alpha^2 - \beta^2) - 16\alpha^2 (\Lambda^{\phi_{2-}})^2 + \sin \varphi \times \right. \\ \left(-41 (C_{\phi_{1-}})^2 - 200q\alpha C_{\phi_{1-}} \Lambda^{\phi_{2-}} - 8 \left(21C_{\theta_-}(\alpha^2 - \beta^2) - 5 (C_{\phi_{2-}})^2 \right. \right. \\ \left. \left. + 24q\alpha C_{\phi_{2-}} \Lambda^{\phi_{1-}} - 2q^2\alpha^2 \left((12\Lambda^{\phi_{1-}})^2 - 13 (\Lambda^{\phi_{2-}})^2 \right) \right) \right) + \sin \varphi \times \\ \left(4 \left(25 (C_{\phi_{1-}})^2 + 4 (C_{\theta_-}(\alpha^2 - \beta^2) - 6C_{\phi_{2-}} (C_{\phi_{2-}} - 2q\alpha\Lambda^{\phi_{1-}}) \right. \right. \\ \left. \left. + 52q\alpha C_{\phi_{1-}} \Lambda^{\phi_{2-}} \right) - \sin \varphi \left(78 (C_{\phi_{1-}})^2 - 8C_{\phi_{2-}} (9C_{\phi_{2-}} - 4q\alpha\Lambda^{\phi_{1-}}) \right. \right. \\ \left. \left. \left. + 40q\alpha C_{\phi_{1-}} \Lambda^{\phi_{2-}} + \sin \varphi \left((C_{\phi_{1-}})^2 \sin \varphi - 20 (C_{\phi_{1-}})^2 + 16 (C_{\phi_{2-}})^2 \right) \right) \right) \right),$$

$$\begin{aligned}
a_8 = & -4(1 - \sin \varphi)^2 \left(C_{\theta_-}(\alpha^2 - \beta^2) + \alpha^2 \left(13 (\Lambda^{\phi_{1-}})^2 - 14 (\Lambda^{\phi_{2-}})^2 \right) \right. \\
& - 2\alpha^2 \left(37 (\Lambda^{\phi_{1-}})^2 - 38 (\Lambda^{\phi_{2-}})^2 \right) \csc \varphi \\
& + \alpha^2 \left(85 (\Lambda^{\phi_{1-}})^2 - 86 (\Lambda^{\phi_{2-}})^2 \right) \csc^2 \varphi \\
& \left. + \frac{1}{2} C_{\theta_-}(\alpha^2 - \beta^2) (1 - \cos 2\varphi - 4 \sin \varphi) \right), \\
\end{aligned}$$

$$a_9 = 16\alpha^2 \left((\Lambda^{\phi_{1-}})^2 - (\Lambda^{\phi_{2-}})^2 \right) \csc^2 \varphi (1 - \sin \varphi)^3 (7 - 3 \sin \varphi),$$

$$a_{10} = -16\alpha^2 \left((\Lambda^{\phi_{1-}})^2 - (\Lambda^{\phi_{2-}})^2 \right) \csc^2 \varphi (1 - \sin \varphi)^4.$$

Since we want the variable $\gamma = \cos^2 \frac{\theta_-}{2}$ to have *maximum*, the coefficient a_{10} must be negative, i.e.

$$(\Lambda^{\phi_{1-}})^2 > (\Lambda^{\phi_{2-}})^2.$$

Taking this into account, we rewrite $\sum_{i=0}^{10} a_i \gamma^i$ as

$$\begin{aligned}
\sum_{i=0}^{10} a_i \gamma^i &= -a_{10} \left(-\gamma^{10} - \frac{1}{a_{10}} \sum_{j=0}^9 a_j \gamma^j \right) \equiv -a_{10} \left(-\gamma^{10} - \sum_{j=0}^9 B_j \gamma^j \right) \\
&= -a_{10}(\gamma_1 - \gamma) \prod_{k=2}^{10} (\gamma - \gamma_k),
\end{aligned}$$

where $\gamma_1 = \gamma_{max}$ and

$$-B_0 = -\prod_{k=1}^{10} \gamma_k, \dots, -B_9 = \sum_{k=1}^{10} \gamma_k.$$

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